LACK OF CONTACT IN A LUBRICATED SYSTEM

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Abstract. We consider the problem of a rigid surface moving over a flat plane. The surfaces are separated by a small gap filled by a lubricant fluid. The relative position of the surfaces is unknown except for the initial time $t = 0$. The total load applied over the upper surface is a know constant for $t > 0$. The mathematical model consists in a coupled system formed by Reynolds variational inequality for incompressible fluids and Newton′s second Law. In this paper we study the global existence and uniqueness of solutions of the evolution problem when the position of the surface presents only one degree of freedom, under extra assumptions on its geometry. The existence of steady states is also studied.

Key words: lubricated systems, Reynolds variational inequality, global solutions, sationnary solutions.
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1. Introduction

Lubricated contacts are widely used in mechanical systems to connect solid bodies that are in relative motion. A lubricant fluid is introduced in the narrow space between the bodies with the purpose of avoiding direct solid-to-solid contact.
This contact is said to be in the hydrodynamic regime, and the forces transmitted between the bodies result from the shear and pressure forces developed in the lubricant film.

We consider one of the simplest lubricated systems which consists of two rigid surfaces in hydrodynamic contact. The bottom surface, assumed planar and horizontal moves with a constant horizontal translation velocity and a vertical given force $F > 0$ is applied vertically on the upper body.

The wedge between the two surfaces is filled with an incompressible fluid. We suppose that the wedge satisfy the thin-film hypothesis, so that a Reynolds-type model can be used to describe the problem.

We denote by Ω the two-dimensional domain in which the hydrodynamic contact occurs. We assume that Ω is open, bounded and with regular boundary $\partial\Omega$. Without lost of generality we consider $0 \in \Omega$. We assume that the upper body, the *slider*, is allowed to move only by vertical translation. The normalized distance between the surfaces is given by

$$h(x,t) = h_0(x) + \eta(t)$$

where $\eta(t) > 0$ represents the vertical translation of the slider and $h_0 : \Omega \longrightarrow [0, \infty[$ describes the shape of the slider and is a given function satisfying

$$h_0 \in C^1(\bar{\Omega}), \quad \min_{x\in\Omega} h_0(x) = h_0(0) = 0. \tag{1.1}$$

The mathematical model we study considers the possible cavitation in the thin film, so the (normalized) pressure "p" of the fluid satisfies the Reynolds variational inequality (see [7]):

$$\int_\Omega h^3 \nabla p \cdot \nabla(\varphi - p) \geq \int_\Omega h \frac{\partial}{\partial x_1}(\varphi - p) - \eta'(t) \int_\Omega (\varphi - p), \quad \forall \varphi \in K \tag{1.2}$$

where

$$K = \left\{ \varphi \in H_0^1(\Omega) : \varphi \geq 0 \right\},$$

and "∇" denotes the gradient with respect to the variables $x \in \Omega$. Without lost of generality we assume the velocity of the bottom surface is oriented in the direction of the x_1 - axis and its normalized value is equal to 1.

The equation of motion of the slider is

$$\eta'' = \int_\Omega p dx - F \qquad \text{(second Newton Law)} \tag{1.3}$$

completed with the initial conditions:

$$\eta(0) = \eta_0 \tag{1.4}$$

$$\eta'(0) = \eta_1, \tag{1.5}$$

where $\eta_0 > 0$, $\eta_1 \in I\!R$ are given data.

The unknowns of the problem are the pressure $p(x,t)$ and the vertical displacement of the slider $\eta(t)$. It is known that for any given C^1 function $\eta(t)$ the problem (1.2) is well posed (see for instance [11]).

The system (1.2)-(1.5) is equivalent to the following Cauchy problem for a second order ordinary differential equation in η:

$$\begin{cases} \eta'' = G(\eta, \eta') \\ \eta'(0) = \eta_1, \\ \eta(0) = \eta_0, \end{cases} \tag{1.6}$$

where $G :]0, \infty[\times I\!R \longrightarrow I\!R$ is given by

$$G(\beta, \gamma) := \int_\Omega q(x)dx - F,$$

and $q \in K$ (depending on β and γ) is the unique solution to

(1.7)

$$\begin{cases} \displaystyle\int_\Omega (h_0 + \beta)^3 \nabla q \cdot \nabla(\varphi - q) \geq \int_\Omega h_0 \frac{\partial}{\partial x_1}(\varphi - q) - \gamma \int_\Omega (\varphi - q) \\ \forall \varphi \in K. \end{cases}$$

The main goal of the paper is to give sufficient conditions on the shape h_0 of the slider to obtain global existence on time to (1.6), i.e. there is no contact solid-to-solid for $t < \infty$. We also study the existence of steady states of the problem. Another interesting physical question which we adress here is to see if there exists a "barrier" value $\eta_b > 0$ such that $\eta(t) \geq \eta_b, \quad \forall\, t > 0$.

We prove the existence of η_b for two of the three cases studied. Third case (the so called "flat case"), we prove that η tends to 0 as $t \to \infty$. The main ideas of these results are the following: when the distance between the surfaces decreases (i.e. $\eta' \leq 0$) there exists a lower bound of the force exerted by the pressure of the fluid on the upper body. This lower bound admits an expression of the form $F_S + F_D$, where F_S is a "spring-like" force and F_D is a "damping force" (see Corollary (3.1) and Remark 3.1).

F_S depends only on the position $\eta(t)$ and represents the force exerted by the pressure of the fluid for the stationary position in an auxiliary sub-domain U of Ω.

F_D is of the form $F_D = -\eta' d$ where d is a "dumping" coefficient and depends only on η. The global existence of the solution η is a consequence of the velocity of blow up of d when η tends to 0. The existence of a "barrier" η_b is based on the fact that F_S blows up when η tends to 0. In the "flat case" the force F_S is equal to zero, which explains the non existence of a barrier.

The present work is related to different articles on the fluid-rigid interaction problems (see for example [4], [5], [8], [9] and [10], for a non-exhaustive bibliography on this subject). These papers concern

the study of the motion of one or many rigid bodies inside a domain $Q \in I\!R^n, n = 2, 3$, filled with an incompressible fluid with constant viscosity. The mathematical model is a coupled system between Navier-Stokes equations modeling the fluid and second Newton Law to describe the rigid bodies positions. A relevant problem in this context is the so called "non-collision" problem, where the question is to know if this body will touch the boundary ∂Q of the fluid in finite time.
In [10] Hillairet consider the particular case where Q is the half-plane $I\!R \times I\!R_+$ and the rigid body is a disk which moves only along the vertical axis. He proves that in absence of external forces the solution is defined globally in time. He also shows that the disk remains all the time "far" from the boundary.
In [8] Gérard-Varet and Hillairet consider a more general shape of the rigid body in a general domain Q in presence of gravity. They prove the existence of a global in time solution of the problem, but now the rigid body can go the boundary of the domain as t goes to infinity. Similar results are given by Hesla in [9].

The main difference between the above mentioned works and the present one is the obtention in this study of a "barrier" value $\eta_b > 0$ for any exterior force F. We can explain this difference by the high shear and pressure that develop in a lubricant fluid film, due especially to the relative motion of the closed surfaces. An interesting open question is to see if similar "barrier" results can be obtained for situations when the thin film hypothesis is not satisfied in the fluid (so the full Navier-Stokes equations must be used in the place of Reynolds models), but relative horizontal motion exists between the two surfaces.

Fluid-rigid interaction problems in lubrication where also considered in [6] where Reynolds equation is used in the place of Reynolds variational inequality in the particular "flat" case. We also mention the papers [1], [2] and [3], where the existence of steady states is studied for lubricated systems with two degrees of freedom.

The contents of the paper are the following:
In Section 2 we precise the hypothesis on h_0 and present the main results of the paper. In Section 3 we give some preliminary results and Section 4 is devoted to the proof of the theorems of Section 2.

2. Main results

We begin by the local in time existence and uniqueness result, for which the minimal hypothesis (1.1) is sufficient.

Theorem 2.1. *The function G is locally Lipschitzian, so we have the existence and uniqueness of solution to (1.6) locally in time.*

Let $[0,T[$ be the maximal interval of existence of solution to (1.6), so $\eta \in C^2([0,T[)$.
The main goal of the paper is to prove that $T = +\infty$.
It is equivalent to prove that for any fixed $T > 0$ there exists $m > 0$ and $M > 0$ (depending eventually on T) such that

$$\left\{ \begin{array}{ll} m \leq \eta(t) \leq M, & \text{for all } t \in [0,T[\\ |\eta'(t)| \leq M. & \end{array} \right. \tag{2.8}$$

Moreover, we are interested to know if there exists such constants m and M independent on T.

In order to study the existence of steady states and global existence of solutions to (1.6) we consider three different cases depending on the shape of the slide h_0.

Case I. Line contact

We assume that h is equal to 0 only in the line $\{x_1 = 0\}$ i.e.

$$\left\{ \begin{array}{l} h_0(0,x_2) = 0 \ \text{ for all } x_2 \in I\!R \text{ such that } (0,x_2) \in \Omega \text{ and} \\ h_0(x_1,x_2) > 0 \ \text{ for all } (x_1,x_2) \in \Omega, \ x_1 \neq 0. \end{array} \right.$$

We also assume that there exists $\alpha \geq 1$ such that

$$h_0(x_1,x_2) \sim |x_1|^\alpha \text{ when } x_1 \to 0. \tag{2.9}$$

More precisely there exists a neighborhood W of 0 and a function h_1 regular enough on the closure $\bar{W}$ of W with $h_1 > 0$ on $\bar{W}$ such that

$$h_0(x_1,x_2) = |x_1|^\alpha h_1(x_1,x_2) \quad \text{in} \quad W.$$

Case II. Point contact

We assume that h_0 is equal to 0 only in the point $\{x = 0\}$, i.e.

$$h_0(0) = 0 \text{ and } h_0(x) > 0 \text{ for all } x \in \Omega - \{0\}.$$

We also assume that there exists $\alpha \geq 1$ such that

$$h_0(x) \sim |x|^\alpha \text{ when } x \to 0 \tag{2.10}$$

that is, there exist W and h_1 as in **Case I** such that

$$h_0(x) = |x|^\alpha h_1(x) \quad \text{in} \quad W$$

(where $|\cdot|$ is the euclidian norm in $I\!R^2$).

Case III. Flat slides

We assume that h is flat, i.e.

$$h_0 = 0 \quad \text{on } \ \Omega \tag{2.11}$$

which implies $h(x,t) = \eta(t)$.

The results concerning the existence of steady states for cases I and II are enclosed in the following theorem:

Theorem 2.2. *Let h_0 satisfy assumption 2.9 in case I or 2.10 in case II for α satisfying*

$$\begin{cases} \alpha > 1 & \text{in Case I (line contact)} \\ \alpha > \frac{3}{2} & \text{in Case II (point contact)}\ . \end{cases} \tag{2.12}$$

Then there exists at least one stationary solution $\bar{\eta} > 0$ of the Cauchy problem (1.6)*, i.e.*

$$G(\bar{\eta}, 0) = 0.$$

Remark 2.1. *The problem of uniqueness of the stationary solution is a difficult one. In* [2] *the authors proved the uniqueness of solutions to the 1-dimension problem for a particular function h_0.*

Results of global existence and barrier functions are presented in the following theorem:

Theorem 2.3. *We assume that h_0 satisfy assumption 2.9 in case I or 2.10 in case II for α satisfying*

$$\begin{cases} \alpha \geq \frac{3}{2} & \text{in Case I (line contact)} \\ \alpha \geq 2 & \text{in Case II (point contact)}\ , \end{cases} \tag{2.13}$$

then $T = +\infty$. Moreover, there exist constants m_0, M_0 and M_1 such that $0 < m_0 \leq M_0$ and $M_1 \geq 0$ satisfying $\forall t \geq 0$:

$$\begin{cases} m_0 \leq \eta(t) \leq M_0 \\ |\eta'(t)| \leq M_1, \end{cases}$$

for $t \geq 0$.

Remark 2.2. *For the one-dimensional problem, i.e. Ω is an interval of $I\!R$, the results are the same than in the case II (line contact) for the two-dimensional problem.*

Some relevant questions concerning the dynamical system (1.6) remain open:
- Uniqueness of solution for the steady states.
- Stability of the steady states.
- Existence of periodic solutions.
- The Attractor of the dynamical system.

Theorem 2.4. *We assume that $h_0 \equiv 0$ (Case III), then $T = +\infty$, moreover there exist $M_0, M_1 > 0$ such that*

$$\begin{cases} 0 < \eta(t) \leq M_0 \\ |\eta'(t)| \leq M_1, \end{cases}$$

for $t \geq 0$ and

$$\eta(t) \to 0 \qquad as \qquad t \to +\infty.$$

Moreover there exist $t_0 \geq 0$, $a > 0$ and $b \in I\!R$ with $t_0 + b > 0$, such that

$$\eta(t) \geq \frac{a}{\sqrt{t+b}} \qquad \forall\, t \geq t_0.$$

In addition, no stationary solution exist for the system (1.6).

Remark 2.3. *The same result can be obtained for the corresponding one-dimensional problem.*

3. Some preliminary results on the function G

3.1. **Results for h_0 satisfying** (1.1) **(all cases are included).** In this subsection we proof some preliminary results on G under the minimal hypothesis (1.1). Let V_1 be defined as follows

$$V_1 = \sup_{x\in\Omega} \left\{ -\frac{\partial h_0}{\partial x_1}(x) \right\} \tag{3.14}$$

It is clear that $V_1 \geq 0$.

Lemma 3.1. *i)* *There exists $c_1 > 0$ such that*

$$G(\beta, \gamma) \leq \frac{c_1}{\beta^3} - F \qquad \forall \beta > 0,\ \gamma \geq 0.$$

$$ii) \qquad G(\beta, \gamma) = -F \qquad \forall \beta > 0,\ \gamma \geq V_1.$$

Proof. i) We take $\varphi = 0$ and $\varphi = 2q$ in (1.7) to have

$$\int_\Omega (h_0 + \beta)^3 |\nabla q|^2 = \int_\Omega h_0 \frac{\partial q}{\partial x_1} - \gamma \int_\Omega q.$$

We use the inequalities $h_0 + \beta \geq \beta$, $\gamma \geq 0$, $q \geq 0$ to obtain

$$\beta^3 \int_\Omega |\nabla q|^2 \leq \int_\Omega h_0 \frac{\partial q}{\partial x_1}.$$

We use the Poincaré inequality and the proof of case i) ends.
ii) The inequality (1.7) can be written:

$$\int_\Omega (h_0 + \beta)^3 \nabla q \cdot \nabla(\varphi - q) \geq - \int_\Omega \left(\frac{\partial h_0}{\partial x_1} + \gamma \right) (\varphi - q)\, dx, \quad \forall \varphi \in K.$$

Since

$$\gamma + \frac{\partial h_0}{\partial x_1} \geq 0, \quad \forall\, x \in \Omega$$

we have that $q = 0$ which gives the result. □

The rest of the results enclosed in this section concern the function $G(\beta, \gamma)$ when $\gamma \leq 0$. We begin by a general result on variational inequalities.

Lemma 3.2. *Let $a \in L^\infty(\Omega)$ such that $\inf_\Omega a > 0$. Let $f \in H^{-1}(\Omega)$ and $q \in K$ be the solution of the problem*

$$\int_\Omega a \nabla q \cdot \nabla(\varphi - q) \geq < f, \varphi - q >, \qquad \text{for all } \ \varphi \in K. \tag{3.15}$$

Let $U \subset \Omega$ arbitrary and open, and let $r \in H_0^1(U)$ the solution to

$$\int_U a \nabla r \cdot \nabla \psi = < f, \psi >, \qquad \text{for all } \ \psi \in H_0^1(U). \tag{3.16}$$

Then $q \geq r$ on U.

Proof. We consider $\psi \in H_0^1(U)$, $\psi \geq 0$ arbitrary, and we extend it to Ω by 0 and denote the extended function by $\tilde{\psi}$ which belongs to K. For simplicity we omit the tilde. We take $\varphi = q + \psi$ in (3.15) to obtain

$$\int_U a \nabla q \cdot \nabla \psi \geq < f, \psi >, \tag{3.17}$$

Let us denote $\xi = q - r$. From (3.16) and (3.17) we have

$$\int_U a \nabla \xi \cdot \nabla \psi \geq 0, \tag{3.18}$$

for any $\psi \in H_0^1(U)$, $\psi \geq 0$. On the other hand we have $\xi \geq 0$ on ∂U. From the maximum principle we obtain $\xi \geq 0$ on U which proves the lemma. □

The following result is a consequence of the above lemma and non-negativity of the solution q to (1.7) in Ω.

Corollary 3.1. *Let us denote for any open set $U \subset \Omega$ and any $\beta > 0$ by $q_{1\beta}$ and $q_{2\beta}$ the solutions to the following problems*

$$\begin{cases} -\nabla \cdot [(h_0 + \beta)^3 \nabla q_{1\beta}] = -\frac{\partial h_0}{\partial x_1} & \text{on } U \\ \\ q_{1\beta} = 0, & \text{on } \partial U \end{cases} \tag{3.19}$$

and

$$
(3.20) \qquad \begin{cases} -\nabla \cdot [(h_0+\beta)^3 \nabla q_{2\beta}] = 1 & \text{on } U \\ \\ q_{2\beta} = 0, & \text{on } \partial U \end{cases}
$$

respectively. We then have

$$
(3.21) \qquad G(\beta,\gamma) \geq \int_U q_{1\beta} dx - \gamma \int_U q_{2\beta} dx - F.
$$

for all $\beta > 0$, $\gamma \in I\!R$ and $U \subset \Omega$ open.

Remark 3.1. *The expressions $\int_U q_{1\beta} dx$ and $\int_U q_{2\beta} dx$ represent the force "F_S" and the damping coefficient "d" respectively, as we described in the Introduction.*

3.2. **The case of non-horizontal slider.** In this subsection we assume $h_0 \neq 0$ and also that h_0 satisfies the hypothesis of Cases I or II (line contact and point contact case respectively). We prove the existence of a sub-domain $U \subset \Omega$ such that the averages of the corresponding functions $q_{1\beta}$ and $q_{2\beta}$ are "large" in some sense when β is small.
We denote by $\rho \geq 0$ and $\theta \in [0, 2\pi]$ the polar coordinates of (x_1, x_2).

Lemma 3.3. a) *Case I. (Line contact)*
There exist $\delta, \beta_0, c_2 > 0$ and $B_{l,\beta}$ defined by

$$
B_{l,\beta} :=]-2\beta^{1/\alpha}, -\beta^{1/\alpha}[\times]-\delta, \delta[
$$

such that for any $0 < \beta \leq \beta_0$ we have

$$
\frac{\partial h_0}{\partial x_1} \leq -c_2 \beta^{1-1/\alpha} \qquad \text{on } B_{l,\beta}.
$$

b) *Case II. (Contact point)*
There exists $\theta_0 \in]0, \frac{\pi}{2}[$, c_2, $\beta_0 > 0$ and the sector $B_{p,\beta}$ defined by

$$
B_{p,\beta} = \{(x_1,x_2) \in I\!R^2;\ \beta^{1/\alpha} \leq \rho \leq 2\beta^{1/\alpha};\ \pi - \theta_0 \leq \theta \leq \pi + \theta_0\}
$$

such that for any $0 < \beta \leq \beta_0$:

$$
\frac{\partial h_0}{\partial x_1} \leq -c_2 \beta^{1-1/\alpha} \qquad \text{on } B_{p,\beta}.
$$

Proof. a) We have for $x_1 \leq 0$

$$
\frac{\partial h_0}{\partial x_1} = -\alpha(-x_1)^{\alpha-1} h_1 + (-x_1)^\alpha \frac{\partial h_1}{\partial x_1} = (-x_1)^{\alpha-1} h_1(x) \left[-\alpha - x_1 \frac{\frac{\partial h_1}{\partial x_1}}{h_1(x)} \right].
$$

Since $h_1 > 0$ on $\bar{W}$ we obtain

$$x_1 \frac{\frac{\partial h_1}{\partial x_1}}{h_1(x)} \longrightarrow 0 \quad \text{when } x \longrightarrow 0$$

and the result is obvious.

b) For any x in $W - \{0\}$ we have

$$\frac{\partial h_0}{\partial x_1} = \alpha |x|^{\alpha-1} \frac{x_1}{|x|} h_1 + |x|^\alpha \frac{\partial h_1}{\partial x_1} = |x|^{\alpha-1} h_1 \left(\alpha \frac{x_1}{|x|} + |x| \frac{\frac{\partial h_1}{\partial x_1}}{h_1} \right).$$

Now we can chose $\theta_0 \in]0, \frac{\pi}{2}[$ such that $\frac{x_1}{|x|} < -\frac{1}{2}$ if $\pi - \theta_0 \le \theta \le \pi + \theta_0$ (choose for example $\theta_0 = \frac{\pi}{6}$). On the other hand we have

$$|x| \frac{\frac{\partial h_1}{\partial x_1}}{h_1} \longrightarrow 0 \qquad \text{when} \qquad x \longrightarrow 0$$

which proves the lemma.

□

Lemma 3.4. *Let us consider $q_{1\beta}$, $q_{2\beta}$ the solutions to (3.19)-(3.20) where U is given by*

$$U := B_{l,\beta} \quad \text{in case I,}$$

$$U := B_{p,\beta} \quad \text{in case II,}$$

with $B_{l,\beta}$ and $B_{p,\beta}$ defined in Lemma 3.3. Then there exists β_0, c_3, $c_4 > 0$ such that for any $\beta \in]0, \beta_0]$ we obtain

$$\left\{ \begin{array}{ll} \int_{B_{l,\beta}} q_{1\beta}(x)dx \ge c_3 \beta^{2(1/\alpha-1)} & \text{in case I} \\ \int_{B_{p,\beta}} q_{1\beta}(x)dx \ge c_3 \beta^{3/\alpha-2} & \text{in case II} \end{array} \right. \tag{3.22}$$

moreover

$$\left\{ \begin{array}{ll} \int_{B_{l,\beta}} q_{2\beta}(x)dx \ge c_4 \beta^{3(1/\alpha-1)} & \text{in case I} \\ \int_{B_{p,\beta}} q_{2\beta}(x)dx \ge c_4 \beta^{4/\alpha-3} & \text{in case II.} \end{array} \right. \tag{3.23}$$

Proof. From (3.20) we deduce

$$\int_U q_{2\beta} dx = \int_U (h_0 + \beta)^3 |\nabla q_{2\beta}|^2 dx. \tag{3.24}$$

From the equality

$$\int_U (h_0 + \beta)^3 \nabla q_{2\beta} \cdot \nabla \varphi = \int_U \varphi, \qquad \text{for all } \varphi \in H_0^1(U)$$

and Cauchy-Schwarz inequality, we get

$$\int_U (h_0 + \beta)^3 |\nabla q_{2\beta}|^2 dx \ge \sup_{\varphi \in H_0^1(U),\ \varphi \ne 0} \left[\frac{\left(\int_U \varphi dx \right)^2}{\int_U (h_0 + \beta)^3 |\nabla \varphi|^2} \right]. \tag{3.25}$$

It suffices to find appropriate test functions $\varphi \in H_0^1(U)$, $\varphi \neq 0$ such that the term

$$\frac{(\int_U \varphi dx)^2}{\int_U (h_0+\beta)^3 |\nabla\varphi|^2}$$

is large enough.

Proof of (3.23)

Case I: Line contact. We choose

$$\varphi(x_1,x_2) = \psi_1\left(\frac{x_1}{\beta^{1/\alpha}}\right)\psi_2(x_2)$$

with $\psi_1 \in \mathcal{D}(]-2,-1[)$, $\psi_1 \geq 0$, $\psi_1 \not\equiv 0$ and $\psi_2 \in D(]-\delta_2,\delta_2[)$, $\psi_2 \geq 0$, $\psi_2 \not\equiv 0$. Then

$$\int_{B_{l,\beta}} \varphi\, dx = \int_{-2\beta^{1/\alpha}}^{-\beta^{1/\alpha}} \int_{-\delta_2}^{\delta_2} \psi_1\left(\frac{x_1}{\beta^{1/\alpha}}\right)\psi_2(x_2)dx_1dx_2 =$$

$$\beta^{1/\alpha}\int_{-2}^{-1}\psi_1(y_1)dy_1\int_{-\delta_2}^{\delta_2}\psi_2(x_2)dx_2$$

and

$$\int_{B_{l,\beta}} (h_0+\beta)^3|\nabla\varphi|^2dx =$$

$$\int_{-2\beta^{1/\alpha}}^{-\beta^{1/\alpha}}\int_{-\delta_2}^{\delta_2}(h_0+\beta)^3\left[\frac{1}{\beta^{2/\alpha}}|\psi_1'\left(\frac{x_1}{\beta^{1/\alpha}}\right)\psi_2(x_2)|^2 + |\psi_1\left(\frac{x_1}{\beta^{1/\alpha}}\right)\psi_2'(x_2)|^2dx\right].$$

It is easy to show that

$$\int_{B_{l,\beta}}(h_0+\beta)^3|\nabla\varphi|^2dx \leq c_2'\beta^{3-1/\alpha}$$

where $c_2' > 0$ is a constant independent on β. (3.25) implies

$$\int_{B_{l,\beta}}(h_0+\beta)^3|\nabla q_{2\beta}|^2dx \geq c_3\beta^{3(1/\alpha-1)}$$

and thanks to (3.24) we obtain $(3.23)_1$.

Case II: Contact point. We choose $\varphi(x_1,x_2) = \psi_3(\frac{\rho}{\beta^{1/\alpha}})\psi_4(\theta)$ with $\psi_3 \in \mathcal{D}(]1,2[)$, $\psi_3 \geq 0$, $\psi_3 \not\equiv 0$, and $\psi_4 \in \mathcal{D}(]\pi-\theta_0,\pi+\theta_0[)$, $\psi_4 \geq 0$, $\psi_4 \not\equiv 0$. In polar coordinates, we have

$$\int_{B_{p,\beta}}\varphi\, dx = \int_{\beta^{1/\alpha}}^{2\beta^{1/\alpha}}\int_{\pi-\theta_0}^{\pi+\theta_0}\psi_3\left(\frac{\rho}{\beta^{1/\alpha}}\right)\psi_4(\theta)\rho d\rho d\theta =$$

$$\beta^{2/\alpha}\int_1^2\psi_3(\rho_1)\rho_1 d\rho_1\int_{\pi-\theta_0}^{\pi+\theta_0}\psi_4(\theta)\, d\theta$$

and

$$\int_{B_{p,\beta}} (h_0+\beta)^3 |\nabla\varphi|^2 dx =$$

$$\int_{B_{p,\beta}} (h_0+\beta)^3 \left[\frac{1}{\beta^{2/\alpha}} \left| \psi_3' \left(\frac{\rho}{\beta^{1/\alpha}} \right) \right|^2 |\psi_4(\theta)|^2 + \frac{1}{\rho^2} \left| \psi_3 \left(\frac{\rho}{\beta^{1/\alpha}} \right) \right|^2 |\psi_4'(\theta)|^2 \right].$$

We easily obtain

$$\int_{B_{p,\beta}} (h_0+\beta)^3 |\nabla\varphi|^2 dx \leq c_2' \beta^3,$$

where $c_2' > 0$ is a constant independent of β. From (3.25) we obtain

$$\int_{B_{p,\beta}} (h_0+\beta)^3 |\nabla q_{2\beta}|^2 dx \geq c_3 \beta^{4/\alpha - 3},$$

and by (3.24) we get $(3.23)_2$.

Proof of (3.22)

From lemma 3.3 we have

$$-\frac{\partial h_0}{\partial x_1} \geq c_2 \beta^{1-1/\alpha} \qquad \text{on } B_{l,\beta} \qquad \text{(in case I)}$$

and

$$-\frac{\partial h_0}{\partial x_1} \geq c_2 \beta^{1-1/\alpha} \qquad \text{on } B_{p,\beta} \qquad \text{(in case II)}.$$

By maximum principle we deduce the inequality

$$q_{1\beta} \geq c_2 \beta^{1-1/\alpha} q_{2\beta}, \qquad \text{on } B_{l,\beta} \qquad \text{(respectively } B_{p,\beta}\text{)}.$$

By (3.23) the proof ends. □

The following corollary is a consequence of Corollary 3.1 and Lemma 3.4.

Corollary 3.2. *For any $\beta \in]0, \beta_0]$ and $\gamma \leq 0$ we have*

$$G(\beta,\gamma) \geq c_3 \beta^{2(1/\alpha - 1)} - \gamma c_4 \beta^{3(1/\alpha - 1)} - F, \qquad \textit{in case I}$$

or

$$G(\beta,\gamma) \geq c_3 \beta^{3/\alpha - 2} - \gamma c_4 \beta^{4/\alpha - 3} - F, \qquad \textit{in case II.}$$

with β_0, c_3, c_4 as in Lemma 3.4.

4. Proof of the main results

We consider $\eta(t)$ the solution of the Cauchy problem (1.6) defined on the maximal interval $[0, T[$.

4.1. **Bounds on η for the non-horizontal slider case.** In this subsection we assume that h_0 satisfies the hypothesis of Cases I or II (line contact and point contact case respectively). We prove that η and η' are bounded and η reminds "far" from 0. We first prove in Proposition 4.1 that η' admits an upper bound and the same for η in Proposition 4.2. These results are needed to prove the existence of lower bounds for η' (Propostion 4.3) and η (Proposition 4.4)
Let V_2 be defined by

$$V_2 = \max\{\eta_1 + 1, V_1\} \tag{4.26}$$

for V_1 as in (3.14), then we have:

Proposition 4.1.

$$\eta'(t) < V_2 \qquad \forall t \in [0, T[.$$

Proof. We argue by the contrary and assume that $t_1 > 0$ is the first point such that $\eta'(t_1) = V_2,$ which implies $\eta''(t_1) \geq 0$ which contradicts Lemma 3.1 *ii)* where $\eta''(t_1) = -F$. □

We introduce two energies $E_1, E_2 :]0, +\infty[\times I\!R \to I\!R$ defined by

$$E_1(\beta, \gamma) = \frac{1}{2}\gamma^2 + F\beta$$

and

$$E_2(\beta, \gamma) = \frac{1}{2}\gamma^2 + F\beta + \frac{c_1}{2\beta^2}$$

for c_1 as in Lemma 3.1.
The energies E_1 and E_2 are used in the following lemma when $\eta(t)$ is non-increasing or non-decreasing respectively.

Lemma 4.1. *For any $t \in [0, T[$ we have*

$$i) \qquad \frac{d}{dt}E_1(\eta(t), \eta'(t)) \leq 0 \quad \text{if} \quad \eta'(t) \leq 0$$

$$ii) \qquad \frac{d}{dt}E_2(\eta(t), \eta'(t)) \geq 0 \quad \text{if} \quad \eta'(t) \geq 0.$$

Proof. *i)* We multiplying the equation

$$\eta'' + F = G(\eta, \eta') + F$$

by η' and use the inequality $G(\eta, \eta') + F \geq 0$ to obtain the result.
ii) From Lemma 3.1 *i)* we have

$$\eta'' - \frac{c_1}{\eta^3} + F \leq 0.$$

We multiply by η' to end the proof. □

Let D_1 and D_2 be defined by

$$D_1 := \left(\frac{c_1}{F}\right)^{1/3}$$

and

$$D_2 := 2\max\left\{\eta_0, D_1, \frac{1}{F}\left(\frac{1}{2}\eta_1^2 + F\eta_0 + \frac{c_1}{2\eta_0^2}\right), \frac{1}{F}\left(\frac{1}{2}V_2^2 + FD_1 + \frac{c_1}{2D_1^2}\right)\right\}.$$

Proposition 4.2.

$$\eta(t) < D_2 \qquad \forall t \in [0,T[.$$

Proof. By the contrary we assume that $t_3 > 0$ is the first time such that

$$\eta(t_3) = D_2. \tag{4.27}$$

Then, it results that $\eta'(t_3) > 0$ or $\eta'(t_3) = 0$. In the last case, since $\eta(t_3) > D_1$, Lemma 3.1 *i*) implies

$$\eta''(t_3) = G(\eta(t_3), 0) < 0.$$

So, in both cases, since $\eta \in C^2$, there exists $t_1 \in [0, t_3[$ such that

$$\eta'(t) \geq 0, \quad \forall t \in [t_1, t_3],$$

where t_1 is the smallest number with this property. Two options concerning t_1 are possible:

Option 1: $t_1 = 0$. In this case, we have

$$\eta'(t) \geq 0, \quad \forall t \in [0, t_3].$$

From Lemma 4.1 *ii*) we obtain

$$E_2(\eta(t_3), \eta'(t_3)) \leq E_2(\eta_0, \eta_1)$$

which implies

$$\eta(t_3) \leq \frac{1}{F}\left(\frac{1}{2}\eta_1^2 + F\eta_0 + \frac{c_1}{2\eta_0^2}\right)$$

and contradicts (4.27).

Option 2: $t_1 \in]0, t_3[$.

We have in this case

$$\eta'(t_1) = 0, \ \ \eta'(t) \geq 0 \quad \forall t \in [t_1, t_3]$$

which implies

$$\eta''(t_1) \geq 0.$$

From Lemma 3.1 *i*) we obtain

$$\frac{c_1}{\eta^3(t_1)} \geq F \quad \text{ that is } \quad \eta(t_1) \leq D_1.$$

Let $t_2 \in [t_1, t_3]$ be a time such that

$$\eta(t_2) = D_1.$$

From Lemma 4.1 *ii*) and Proposition 4.1 we have

$$E_2(\eta(t_3), \eta'(t_3)) \leq E_2(\eta(t_2), \eta'(t_2)) \leq \frac{1}{2}V_2^2 + FD_1 + \frac{c_1}{2D_1^2}$$

which implies

$$\eta(t_3) \leq \frac{1}{F}\left(\frac{1}{2}V_2^2 + FD_1 + \frac{c_1}{2D_1^2}\right)$$

and contradicts (4.27) and the proof ends. □

We define V_3 as follows

$$V_3 := \max\left\{1 - \eta_1,\ 2\sqrt{2FD_2},\ 2\sqrt{\eta_1^2 + 2F\eta_0}\right\}. \tag{4.28}$$

Proposition 4.3.

$$\eta'(t) > -V_3, \quad \forall t \in [0, T[.$$

Proof. We argue by the contrary and assume that $t_2 \in]0, T[$ is the first time such that

$$\eta'(t_2) = -V_3. \tag{4.29}$$

We have two options:
Option I. $\eta'(t) \leq 0, \quad \forall t \in [0, t_2]$.
From Lemma 4.1 *i*) we have

$$E_1(\eta(t_2), \eta'(t_2)) \leq E_1(\eta_0, \eta_1)$$

which implies

$$\frac{1}{2}|\eta'(t_2)|^2 \leq \frac{1}{2}\eta_1^2 + F\eta_0$$

and contradicts (4.29).
Option II: There exists $t_1 \in]0, t_2[$ such that

$$\eta'(t_1) = 0 \qquad \text{and} \qquad \eta'(t) \leq 0 \quad \forall t \in [t_1, t_2].$$

Then

$$E_1(\eta(t_2), \eta'(t_2)) \leq E_1(\eta(t_1), 0)$$

which combined with Proposition 4.2 implies

$$\frac{1}{2}|\eta'(t_2)|^2 \leq FD_2$$

and contradicts (4.29). □

The most difficult part is to obtain a lower bound of η (Proposition 4.4). Before we remark that from Corollary 3.2 we have

$$G(\beta,\gamma) \geq c_3\beta^{-s_1} - c_4\gamma\beta^{-1-s_2} - F, \quad \forall \beta \in]0,\beta_0], \quad \forall \gamma \leq 0 \tag{4.30}$$

where β_0, c_3, c_4 were defined in Lemma 3.4 and

$$s_1 = \begin{cases} 2\left(1 - \frac{1}{\alpha}\right) & \text{in Case I (line contact)} \\ 2 - \frac{3}{\alpha} & \text{in Case II (point contact)} \end{cases} \tag{4.31}$$

$$s_2 = \begin{cases} 2 - \frac{3}{\alpha} & \text{in Case I} \\ 2 - \frac{4}{\alpha} & \text{in Case II,} \end{cases} \tag{4.32}$$

Notice that $s_1 > 0$ and $s_2 \geq 0$. Let D_3 and $D_4 > 0$ be defined by

$$D_3 := \frac{2}{3}\min\left\{\eta_0,\ \left(\frac{c_3}{F}\right)^{1/s_1}\right\}, \tag{4.33}$$

(4.34)

$$\begin{cases} D_4 = \frac{1}{2}\min\{\eta_0,\ \beta_0,\ \left(\frac{c_3}{F}\right)^{1/s_1},\ \left(D_3^{-s_2} + \frac{s_2}{c_4}V_3\right)^{-1/s_2}\} & \text{if } s_2 > 0 \\ \text{and} \\ D_4 = \frac{1}{2}\min\{\eta_0,\ \beta_0,\ \left(\frac{c_3}{F}\right)^{1/s_1},\ D_3 e^{-V_3/c_4}\} & \text{if } s_2 = 0. \end{cases}$$

Proposition 4.4. *Under assumption*

$$\begin{cases} \alpha \geq \frac{3}{2} & \textit{in Case I (line contact)} \\ \textit{or} \\ \alpha \geq 2 & \textit{in Case II (point contact)} \end{cases} \tag{4.35}$$

we have $\eta(t) > D_4, \quad \forall t \in [0,T[$.

Proof. By the contrary we assume $t_2 \in]0,T[$ is the first time such that

$$\eta(t_2) = D_4. \tag{4.36}$$

Notice that $\eta_0 > D_3 > D_4$. Let $t_1 \in]0,t_2[$ be the last point where

$$\eta(t_1) = D_3. \tag{4.37}$$

By definition of D_3 we have:

$$\begin{cases} \eta'(t_1) \leq 0, \quad \eta'(t_2) \leq 0 \\ c_3\eta(t)^{-s_1} > F, \quad \forall t \in [t_1,t_2]. \end{cases} \tag{4.38}$$

We first see

$$\eta'(t) \leq 0, \quad \forall t \in [t_1,t_2]. \tag{4.39}$$

Suppose that (4.39) is false, then there exists $\tau \in]t_1,t_2[$ such that

$$\eta'(\tau) > 0.$$

Let τ_1 be the supremum of $\tau \in]t_1, t_2[$ satisfying $\eta'(\tau) > 0$. It is clear that $\tau_1 < t_2$ and it is a local maximum of η which implies

$$\begin{cases} \eta'(\tau_1) = 0 \\ \eta''(\tau_1) \leq 0. \end{cases} \tag{4.40}$$

Then from (4.30) and (4.38) we have

$$\eta''(\tau_1) = G(\eta(\tau_1), 0) \geq \frac{c_3}{\eta(\tau_1)^{s_1}} - F > 0$$

which contradicts (4.40). Then (4.39) is proved.
Combining (4.30) and (4.38) we deduce

$$\eta'' \geq -c_4 \eta' \eta^{-1-s_2} \quad \text{on} \ \ [t_1, t_2]. \tag{4.41}$$

Case i**):** $s_2 > 0$.
We integrate (4.41) to deduce

$$\eta'(t) \geq \eta'(t_1) + \frac{c_4}{s_2}\eta(t)^{-s_2} - \frac{c_4}{s_2}\eta(t_1)^{-s_2}, \quad \forall t \in [t_1, t_2]$$

and thanks to (4.39) and Proposition 4.3 applied for $t = t_1$ we obtain

$$\frac{c_4}{s_2}\eta(t_2)^{-s_2} \leq \frac{c_4}{s_2}\eta(t_1)^{-s_2} + V_3.$$

Since $\eta(t_1) = D_3$ it results

$$\eta(t_2) \geq \left(D_3^{-s_2} + \frac{s_2}{c_4} V_3 \right)^{-1/s_2}$$

which contradicts (4.36).
Case ii**):** $s_2 = 0$.
We integrate (4.41) to obtain

$$\eta'(t) \geq \eta'(t_1) + c_4 \log\left(\frac{1}{\eta(t)}\right) - c_4 \log\left(\frac{1}{\eta(t_1)}\right), \quad \forall t \in [t_1, t_2]$$

which implies

$$c_4 \log\left(\frac{1}{\eta(t_2)}\right) \leq c_4 \log\left(\frac{1}{\eta(t_1)}\right) + V_3.$$

Then

$$\eta(t_2) \geq D_3 e^{-V_3/c_4}$$

which contradicts (4.36). $\square$

4.2. **Bounds on η for the flat case.** We consider the case $h_0 \equiv 0$. Let us introduce the auxiliary function w defined as the unique solution to the problem

$$\begin{cases} -\Delta w = 1 & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{4.42}$$

and define the constant $C(\Omega)$ by

$$C(\Omega) = \int_\Omega w(x)\,dx.$$

By maximum principle we have $w > 0$ on Ω which implies

$$C(\Omega) > 0.$$

In the following, for any real number z we denote $z^+ = \max\{z, 0\}$ (positive part) and $z^- = -\min\{z, 0\}$ (negative part). We have the identity $z = z^+ - z^-$.

Lemma 4.2. *η satisfies the following differential equation*

$$\eta'' = C(\Omega)\frac{(\eta')^-}{\eta^3} - F.$$

Proof. For $h_0 \equiv 0$ the inequality (1.7) becomes

$$\beta^3 \int_\Omega \nabla q \cdot \nabla(\varphi - q) \geq -\gamma \int_\Omega (\varphi - q), \quad \forall\, \varphi \in K. \tag{4.43}$$

The required result is a direct consequence of the following facts
- if $\gamma \geq 0$ the solution of (4.43) is $q = 0$
- if $\gamma < 0$ the solution of (4.43) is $q = -\frac{\gamma w}{\beta^3}$. □

The bounds on η and η' can be summarized in the following proposition

Proposition 4.5. *The following inequalities are valid:*
I) *For $\eta_1 \leq 0$ and $t \in]0, T[$ we have*
Ia) $-\sqrt{\eta_1^2 + 2F\eta_0} \leq \eta'(t) < 0$
Ib) $\eta_0 \left[\frac{C(\Omega)}{C(\Omega) + 2\eta_0^2 Ft - 2\eta_0^2\eta_1}\right]^{1/2} \leq \eta(t) \leq \eta_0$.
II) *For any $\eta_1 > 0$ we define $t_0 = \frac{\eta_1}{F}$ and $\hat\eta_0 = \eta_0 + \frac{\eta_1^2}{2F}$, then $t_0 < T$ and we have*
IIa) $\eta(t) = -\frac{1}{2}Ft^2 + \eta_1 t + \eta_0$ *for* $t \in [0, t_0]$
IIb) $-\sqrt{2F\hat\eta_0} \leq \eta'(t) < 0$ *for* $t \in]t_0, T[$
IIc) $\hat\eta_0 \left[\frac{C(\Omega)}{C(\Omega) + 2\hat\eta_0^2 F(t - t_0)}\right]^{1/2} \leq \eta(t) \leq \hat\eta_0$ *for* $t \in]t_0, T[$.

Proof. **I)** We assume that $\eta_1 \leq 0$. Then, Lemma 4.2 implies $\eta''(0) = -F$ and therefore there exists a point $t_1 \in]0, T]$ such that

$$\eta'(t) < 0, \quad \forall\, t \in]0, t_1[,$$

where t_1 denotes the largest element with this property. We now prove

$$\eta'(t) < 0, \quad \forall\, t \in]0, T[, \tag{4.44}$$

which is equivalent to assert $t_1 = T$.
In order to prove (4.44) we argue by contradiction and assume $t_1 < T$ and $\eta'(t_1) = 0$ which implies $\eta''(t_1) \geq 0$ and contradicts $\eta''(t_1) = -F$ which is obtained from Lemma 4.2 and proves (4.44). (4.44) implies

$$\eta'' = -C(\Omega)\frac{\eta'}{\eta^3} - F \quad \text{ on } \quad [0, T[. \tag{4.45}$$

We multiply by η' to obtain that $\frac{1}{2}(\eta')^2 + F\eta$ is a non-increasing function on $[0, T]$. This completes the proof of the double inequality in **Ia)**. Since η is a non-increasing function, we deduce the inequality of the right-hand side of **Ib)**. Now we integrate (4.2) over $[0, t[$ to obtain

$$\eta' = \eta_1 + \frac{C(\Omega)}{2\eta^2} - \frac{C(\Omega)}{2\eta_0^2} - Ft.$$

Thanks to $\eta' < 0$ of **Ia)** we obtain

$$\frac{C(\Omega)}{2\eta^2} < \frac{C(\Omega)}{2\eta_0^2} + Ft - \eta_1 \quad \text{ on } \quad]0, T[$$

which completes the proof of **Ib)**.
II) we assume that $\eta_1 > 0$. Then, Lemma 4.2 implies $\eta'' = -F$ for $t \in [0, t_0]$ which proves **IIa)**. Since $\eta'(t_0) = 0$ and $\eta(t_0) = \hat{\eta}_0$ the proofs of **IIb)** and **IIc)** are similar to the proofs of **Ia)** and **Ib)** respectively. □

4.3. **Proofs of the theorems.** *Proof of Theorem 2.1.* Let us fix $\beta > 0$ and $\gamma \in I\!R$ and take $\tilde{\beta} > 0$ and $\tilde{\gamma} \in I\!R$ such that $(\tilde{\beta}, \tilde{\gamma})$ are close enough to (β, γ). We denote by $\tilde{q} \in K$ the solution to the Reynolds inequality

$$\int_\Omega \left(h_0 + \tilde{\beta}\right)^3 \nabla\tilde{q} \cdot \nabla(\varphi - \tilde{q}) \geq \int_\Omega h_0 \frac{\partial}{\partial x_1}(\varphi - \tilde{q}) - \tilde{\gamma}\int_\Omega (\varphi - \tilde{q}), \quad \forall \varphi \in K$$

which can be written in the form

$$\begin{aligned} \int_\Omega (h_0 + \beta)^3 \nabla\tilde{q} \cdot \nabla(\varphi - \tilde{q}) \geq \int_\Omega h_0 \frac{\partial}{\partial x_1}(\varphi - \tilde{q}) - \tilde{\gamma}\int_\Omega (\varphi - \tilde{q}) + \\ + (\beta - \tilde{\beta})\int_\Omega A_{\beta,\tilde{\beta}}(x)\nabla\tilde{q} \cdot \nabla(\varphi - \tilde{q}) \end{aligned} \tag{4.46}$$

where $A_{\beta,\tilde{\beta}}$ is uniformly bounded in $\tilde{\beta}$.
We take $\varphi = \tilde{q}$ in (1.7), $\varphi = q$ in (4.46) and we add both inequalities to get

$$\int_\Omega (h_0+\beta)^3 \,|\nabla(\tilde{q}-q)|^2 \leq |\tilde{\gamma}-\gamma||\Omega|^{1/2}\|\tilde{q}-q\|_{L^2(\Omega)} + \\ + |\tilde{\beta}-\beta|\|A_{\beta,\tilde{\beta}}\|_{L^\infty(\Omega)}\|\nabla\tilde{q}\|_{L^2(\Omega)}\|\nabla(\tilde{q}-q)\|_{L^2(\Omega)}.$$

By Poincaré inequality the proof ends.
Proof of Theorem 2.2.
Let us introduce the function $g :]0,+\infty[\to I\!R$ defined by

$$g(\beta) = G(\beta, 0).$$

From Theorem 2.1, it is clear that g is continuous. Lemma 3.1 $i)$ and Corollary 3.2 imply

$$\lim_{\beta\to+\infty} g(\beta) = -F \quad \text{and} \quad \lim_{\beta\to 0} g(\beta) = +\infty$$

which proves the theorem.
Proof of Theorem 2.3.
The result is a consequence of Propositions 4.1, 4.2, 4.3 and 4.4.
Proof of Theorem 2.4.
The non-existence of stationary solutions comes from Lemma 4.2. The results concerning the evolution on the position follow from Proposition 4.5.

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